Tutorial – 02

A sensor is connected to a preamplifier featuring a wide bandwidth, limited by a single pole at f_S=100kHz. The acquisition system provides a single pulse with a known shape and the pulse amplitude V_P is to be measured. An auxiliary synchronism signal is available, which points out the arrival time of the signal. The noise coming with the signal features a uniform **unilateral** spectral density equal to $\sqrt{S_{N,U}}$ = 10nV/$\sqrt{Hz}$ and a wide band, limited by the preamplifier.

Consider the following two cases:

- rectangular signal with a pulse duration T_P=10ms
- exponential signal V_P1(t)e^-(t/T_P), with T_P=10ms

<u>For both cases:</u>

a) Evaluate the minimum measurable amplitude for each pulse without any filtering, i.e. measured directly at the preamp output.
b) <u>Select an **analog** filter</u> to improve the sensitivity of the measurement setup, i.e. a **lower** value of the minimum signal that can be measured. The filter can be different in the two cases. Explain in detail the guidelines to optimize the selection of filter parameters. Select the filter parameters for maximizing the Signal-to-Noise ratio (S/N) and evaluate the minimum measurable amplitude V_P,min.
c) Instead of using an analog filter, consider now an acquisition chain that performs a sampling of the signal and converts the samples into digital words which are sent to a PC for elaboration. Discuss the criteria to select the sampling frequency and what kind of elaboration you would select to optimize the measurement. Select the filter parameters for maximizing the Signal-to-Noise ratio (S/N) and evaluate the minimum measurable amplitude V_P,min.
d) Discuss the advantages and the disadvantages of aforementioned analog and digital filtering solutions. Make a comparison between the two filters with a particular focus on the minimum signal that can be measured.

A) The minimum measurable amplitude for each pulse is the amplitude for a unitary SNR, assuming a measurement at the peak of the signal (possible because we have a sync signal) we can write:

$$SNR = \frac{s}{\sqrt{\sigma_n^2}} = \frac{V_P}{\sqrt{\int_{-\infty}^{\infty} \frac{S_{N,U}}{2} |H_A(f)|^2 df}} = 1 \rightarrow V_{P,min} = \sqrt{\int_{-\infty}^{\infty} \frac{S_{N,U}}{2} |H_A(f)|^2 df}$$

Using the Equivalent Noise BandWidth (ENBW) of a single pole response, we obtain:

$$ENBW = \frac{\pi}{2} f_s \rightarrow \int_{-\infty}^{\infty} \frac{S_{N,U}}{2} |H_A(f)|^2 df = S_{N,U} \int_0^{\frac{\pi}{2} f_s} df = S_{N,U} \frac{\pi}{2} f_s \rightarrow V_{P,min} = \sqrt{S_{N,U} \frac{\pi}{2} f_s} \cong 3.96\ \mu V$$

We can define a bilateral power spectral density:

$$S_{N,B} = \frac{S_{N,U}}{2} = 5 \frac{nV}{\sqrt{Hz}}$$

B) To filter the signals, we can use a Gated Integrator (GI), or a simple RC:

Gated Integrator:

Assuming a GI weight function wide $\boldsymbol{T_G}$ and with an amplitude $\boldsymbol{A}$ starting at the beginning of the signal to sample:

$$\boldsymbol{h(t) = A \cdot \mathbf{rect}_{T_G}\left(t - \frac{T_G}{2}\right) \rightarrow w_m(t) = A \cdot \mathbf{rect}_{T_G}\left(t_m + \frac{T_G}{2} - t\right) \quad with \quad t_m = T_P}$$

Given the auto-correlation of the noise is much shorter (in time) than the filter's one, we can approximate $\boldsymbol{K^E_{w_1w_1}(\tau)}$ as a $\boldsymbol{K^E_{w_1w_1}(0)}$, thus, we can express the noise as:

$$\boldsymbol{\sigma_n^2 = \int_{-\infty}^{\infty} R_{xx}(\tau) K^E_{w_1w_1}(\tau) d\tau \cong K^E_{w_1w_1}(0) \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau = A^2 T_G \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau = A^2 T_G \cdot S_{N,B}}$$

This calculation holds true for any shape of the input signal, alternatively, we could have derived the noise in the frequency domain:

$$\boldsymbol{\sigma_n^2 = \int_{-\infty}^{\infty} S_{N,B} |W_m(f)|^2 \, df = S_{N,B} \int_{-\infty}^{\infty} |W_m(f)|^2 df = S_{N,B} \cdot A^2 T_G^2 \int_0^{f_n} df = S_{N,B} \cdot A^2 T_G^2 f_n = A^2 T_G \cdot S_{N,B}}$$

Gated Integrator and rectangular pulse:

The signal acquired for a rectangular pulse is:

$$\boldsymbol{s = \int_{-\infty}^{\infty} x(t) w_m(t) dt = \int_{-\infty}^{\infty} V_P \mathbf{rect}_{T_P}\left(t - \frac{T_P}{2}\right) A \mathbf{rect}_{T_G}\left(T_P + \frac{T_G}{2} - t\right) dt = \int_0^{T_G} A V_P dt \rightarrow}$$

$$\boldsymbol{s = A T_G \cdot V_P \quad for \quad 0 \leq T_G \leq T_P}$$

The minimum measurable amplitude is the amplitude to achieve a unitary SNR, thus:

$$\boldsymbol{SNR = \frac{s}{\sqrt{\sigma_n^2}} = \frac{A T_G \cdot V_P}{\sqrt{A^2 T_G \cdot S_{N,B}}} = V_P \frac{\sqrt{T_G}}{\sqrt{S_{N,B}}} = 1 \rightarrow V_{P,min} = \sqrt{\frac{S_{N,B}}{T_G}}}$$

Using the optimal value for the acquisition time $\boldsymbol{T_G = T_P}$ we obtain a minimum measurable pulse of:

$$\boldsymbol{V_{P,min} = \sqrt{\frac{S_{N,B}}{T_G}} \cong 70.7 \, nV}$$

Gated Integrator and exponential pulse:

The signal acquired for an exponential pulse is:

$$\boldsymbol{s = \int_{-\infty}^{\infty} x(t) w_m(t) dt = \int_0^{T_G} A V_P e^{-\frac{t}{T_P}} dt = A T_P \cdot V_P \left(1 - e^{-\frac{T_G}{T_P}}\right)}$$

The minimum measurable amplitude for each pulse is the amplitude for a unitary SNR, thus:

$$\boldsymbol{SNR = \frac{s}{\sqrt{\sigma_n^2}} = \frac{A T_P \cdot V_P \left(1 - e^{-\frac{T_G}{T_P}}\right)}{\sqrt{A^2 T_G \cdot S_{N,B}}} = V_P \left(1 - e^{-1}\right) \sqrt{\frac{T_P^2}{S_{N,B} T_G}} = 1 \rightarrow V_{P,min} = \frac{1}{1 - e^{-\frac{T_G}{T_P}}} \sqrt{S_{N,B} \frac{T_G}{T_P^2}}}$$

It is possible to obtain numerically the optimal integration time to maximize the SNR, the condition is $\boldsymbol{T_G \cong \frac{5}{4} T_P}$, giving us a minimum measurable amplitude of:

$$\boldsymbol{V_{P,min} = \frac{1}{1 - e^{-\frac{5}{4}}} \sqrt{\frac{5}{4} \frac{S_{N,B}}{T_P}} \cong 110.8 \, nV}$$

RC filter:

Assuming an RC Low-Pass filter with an exponential weight function with a time constant $\boldsymbol{T_F}$, starting at the beginning of the signal to sample:

$$h(t)=\frac{1}{T_F}u(t)e^{-\frac{t}{T_F}} \to w_m(t)=\frac{1}{T_F}u(t_m-t)e^{-\frac{t_m-t}{T_F}} \quad with \quad t_m=T_P$$

Given the auto-correlation of the noise is much shorter (in time) than the filter's one, we can approximate $\boldsymbol{K^E_{w_1w_1}(\tau)}$ as $\boldsymbol{K^E_{w_1w_1}(0)}$, thus, we can express the noise as:

$$\sigma_n^2=\int_{-\infty}^{\infty}R_{xx}(\tau)K^E_{w_1w_1}(\tau)d\tau \cong K^E_{w_1w_1}(0)\int_{-\infty}^{\infty}R_{xx}(\tau)d\tau=\frac{1}{2T_F}\int_{-\infty}^{\infty}R_{xx}(\tau)d\tau=\frac{1}{2T_F}\cdot S_{N,B}$$

Alternatively, we can calculate the same result in the frequency domain:

$$\sigma_n^2=\int_{-\infty}^{\infty}S_x^P(f)|W_m(f)|^2df=S_{N,B}\int_{-\infty}^{\infty}|W_m(f)|^2df=S_{N,B}\int_0^{f_n}df=f_n\cdot S_{N,B}=\frac{1}{2T_F}\cdot S_{N,B}$$

RC filter and rectangular pulse:

The signal acquired for a rectangular pulse is:

$$s=\int_{-\infty}^{\infty}x(t)w_m(t)dt=\int_{-\infty}^{\infty}\frac{V_P}{\mathrm{T_F}}\mathrm{rect}_{\mathrm{T_P}}\left(t-\frac{T_P}{2}\right)u(T_P-t)e^{-\frac{T_P-t}{T_F}}df=\frac{V_Pe^{-\frac{T_P}{T_F}}}{T_F}\int_0^{T_P}e^{\frac{t}{T_F}}df=V_P\left[1-e^{-\frac{T_P}{T_F}}\right]$$

The minimum measurable amplitude for each pulse is the amplitude for a unitary SNR, thus:

$$SNR=\frac{s}{\sqrt{\sigma_n^2}}=\frac{V_P\left[1-e^{-\frac{T_P}{T_F}}\right]}{\sqrt{\frac{1}{2T_F}\cdot S_{N,B}}}=\frac{V_P}{\sqrt{S_{N,B}}}\sqrt{2T_F}\left[1-e^{-\frac{T_P}{T_F}}\right]=1\to V_{P,min}=\frac{1}{1-e^{-\frac{T_P}{T_F}}}\sqrt{\frac{S_{N,B}}{2T_F}}$$

It is possible to obtain numerically the optimal integration time to maximize the SNR, the condition is $\boldsymbol{T_F\cong\frac{4}{5}T_P}$, giving us a minimum measurable amplitude of:

$$V_{P,min}=\frac{1}{1-e^{-\frac{5}{4}}}\sqrt{\frac{5}{8}\frac{S_{N,B}}{T_P}}\cong 78.3\ nV$$

RC filter and exponential pulse:

The signal acquired for an exponential pulse is:

$$s=\int_{-\infty}^{\infty}x(t)w_m(t)dt=\int_{-\infty}^{\infty}\frac{V_P}{T_F}u(t)e^{-\frac{t}{T_P}}u(t_m-t)e^{-\frac{t_m-t}{T_F}}dt=\frac{V_P}{T_F}e^{-\frac{t_m}{T_F}}\int_0^{t_m}e^{t\frac{(T_P-T_F)}{T_PT_F}}dt$$

Alternatively, in the frequency domain, we have:

$$s=\mathcal{L}^{-1}[S]=\mathcal{L}^{-1}[X(s)W(s)]=\mathcal{L}^{-1}\left[\frac{V_PT_P}{1+sT_P}\cdot\frac{1}{1+sT_F}\right]$$

The result of the integral varies as we chose the value of $\boldsymbol{T_F}$, if we chose $\boldsymbol{T_P\neq T_F}$ the solution is:

$$s=\frac{V_P}{T_F}e^{-\frac{t_m}{T_F}}\left[\frac{T_PT_F}{T_P-T_F}e^{t\frac{(T_P-T_F)}{T_PT_F}}\right]_0^{T_P}=\frac{V_PT_Pe^{-\frac{t_m}{T_F}}}{T_P-T_F}\left[e^{\frac{(T_P-T_F)}{T_F}}-1\right]=\frac{V_PT_Pe^{-\frac{t_m}{T_F}}}{T_P-T_F}=\mathcal{L}^{-1}\left[\frac{V_PT_P}{1+sT_P}\cdot\frac{1}{1+sT_F}\right]$$

If we chose $\boldsymbol{T_P=T_F}$ the solution is:

$$s=\frac{V_P}{T_F}e^{-\frac{t_m}{T_F}}\int_0^{t_m}dt=\frac{V_P}{T_F}t_me^{-\frac{t_m}{T_F}}=\mathcal{L}^{-1}\left[\frac{V_PT_F}{(1+sT_F)^2}\right]$$

We will use $\boldsymbol{T_P=T_F}$, to choose the optimal $\boldsymbol{t_m}$ we must find the maximum of $\boldsymbol{s}$:

$$\frac{\partial s}{\partial t_m}=\frac{V_P}{T_F}e^{-\frac{t_m}{T_F}}-\frac{V_P}{T_F^2}t_m=0\to T_Fe^{-\frac{t_m}{T_F}}=t_m\to t_m=T_F\to\ s=V_Pe^{-1}$$

The minimum measurable amplitude for each pulse is the amplitude for a unitary SNR, thus:

$$SNR=\frac{s}{\sqrt{\sigma_n^2}}=\frac{V_Pe^{-1}}{\sqrt{\frac{1}{2T_F}\cdot S_{N,B}}}=V_Pe^{-1}\sqrt{\frac{2T_F}{S_{N,B}}}=1\to V_{P,min}=e\sqrt{\frac{S_{N,B}}{2T_P}}\cong 136\ nV$$

C) To filter the signals, we can use a Discrete Time Integrator (DTI):

Constant DTI and rectangular pulse:

Assuming a constant weight w_k , and a sampling frequency f_S:

$$w_k = 1$$

The signal acquired for a rectangular pulse is:

$$s = \sum_{k=0}^{N} w_k x_k = \sum_{k=0}^{N} w_k V_P = N V_P$$

The noise acquired considering an uncorrelated input noise is:

$$\sigma_n^2 = \sum_{k=0}^{N} w_k^2 x_k^2 = N \sum_{k=0}^{N} x_k^2 = N \sigma_{n,in}^2$$

The minimum measurable amplitude is the amplitude to achieve a unitary SNR, thus:

$$SNR = \frac{s}{\sqrt{\sigma_n^2}} = \frac{N V_P}{\sqrt{N \sigma_{n,in}^2}} = \frac{V_P}{\sqrt{\sigma_{n,in}^2}} \sqrt{N} = 1 \rightarrow V_{P,min} = \frac{V_{P,min,0}}{\sqrt{N}}$$

If we chose $T_S = 10\,\tau = 15.9\,\mu s$:

$$N = \frac{T_P}{T_S} \cong 629 \rightarrow V_{P,min} = \frac{V_{P,min,0}}{\sqrt{N}} \cong 158\,nV$$

Exponential decaying DTI and exponential pulse:

Assuming a constant weight w_k , and a sampling frequency f_S:

$$w_k = r^k = e^{-k\frac{T_S}{T_P}} \quad with \quad (1 - r) \ll r$$

The signal acquired for an exponential pulse is:

$$s = \sum_{k=0}^{N} w_k x_k = \sum_{k=0}^{N} V_P e^{-k\frac{T_S}{T_P}} A e^{-k\frac{T_S}{T_P}} = A V_P \sum_{k=0}^{N} e^{-2k\frac{T_S}{T_P}} = \sum_{k=0}^{N} \alpha^{-2k} = \frac{A V_P}{1 - \alpha^2} \quad with \quad \alpha = e^{-\frac{T_S}{T_P}} \cong 1 - \frac{2T_S}{T_P}$$

The noise acquired considering an uncorrelated input noise is:

$$\sigma_n^2 = \sum_{k=0}^{N} w_k^2 x_k^2 = \sum_{k=0}^{N} A^2 e^{-2k\frac{T_S}{T_P}} x_k^2 = A^2 \sigma_{n,in}^2 \sum_{k=0}^{N} \alpha^{-2k} = \frac{A^2 \sigma_{n,in}^2}{1 - \alpha^2} \quad with \quad \alpha = e^{-\frac{T_S}{T_P}} \cong 1 - \frac{2T_S}{T_P}$$

The minimum measurable amplitude is the amplitude to achieve a unitary SNR, thus:

$$SNR = \frac{s}{\sqrt{\sigma_n^2}} = \frac{\frac{A V_P}{1 - \alpha^2}}{\sqrt{\frac{A^2 \sigma_{n,in}^2}{1 - \alpha^2}}} = \frac{V_P}{\sqrt{\sigma_{n,in}^2}} \frac{1}{\sqrt{1 - \alpha^2}} \cong \frac{V_P}{\sqrt{\sigma_{n,in}^2}} \sqrt{\frac{T_P}{2T_S}} = 1 \rightarrow V_{P,min} = V_{P,min,0} \sqrt{\frac{2T_S}{T_P}}$$

If we chose $T_S = 10\,\tau = 15.9\,\mu s$:

$$V_{P,min} = V_{P,min,0} \sqrt{\frac{2T_S}{T_P}} \cong 223\,nV$$

D) The Continuous Time (CT) filters are better than their relative Discrete Time (DTI) filters, the best SNR of all the filters analysed is achieved with the Gated Integrator (for both the rectangular and the exponential pulse).
The DTI’s advantage over the CT filter is the possibility to implement weight functions impossible to implement in CT filters, allowing us to implement the optimum weight function.